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NATURAL VIBRATIONS OF A CLAMPED CIRCULAR PLATE WITH RECTILINEAR ORTHOTROPY BY LEAST-SQUARES COLLOCATION

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Abstract—The natural frequencies and mode shapes of a clamped circular plate with rectilinear orthotropy are determined by a modified application of the interior collocation method. In this version of the collocation method, the number of collocation stations is greater than the number of approximation functions. A least-squares error fit is then used to generate the governing eigenvalue problem. Overcollocation has the benefit of minimizing the effect of poor choice of collocation stations to allow for better results consistent with the number and quality of approximation functions employed. Parametric studies are presented to illustrate the effect of orthotropy on the natural vibration data of the clamped circular plate.

INTRODUCTION

In the natural vibration analysis of thin plates, exact solutions occur only for a handful of cases with simple plan forms, amenable boundary conditions and extensional and flexural stiffnesses that appear to be at most orthotropic. For all other cases, approximate analysis methods are employed. The finite element method is acknowledged as the most versatile of these methods. Natural frequencies and mode shapes of a finite element model can be extracted by very efficient eigensolution schemes even for large algebraic eigensystems. The more important approximate analysis techniques antecedent to finite elements include Rayleigh-Ritz, Galerkin and collocation methods. These methods because of various limitations are appropriate for particular problems only. However, whenever they can be used, there is a decided advantage in them over finite elements because substantially less computational effort is required. For data on the vibration of plates, the reader is referred to Leissa[1–6] and Bert[7–10], whose comprehensive bibliographies are invaluable reference sources.

Herein, a natural vibration analysis is given for a clamped circular plate with rectilinear orthotropy. A superficial examination of the problem details will immediately reveal the formidable if not insurmountable difficulties in constructing a closed-form exact solution. Lekhnitskii[11] gave a Rayleigh quotient estimate of the fundamental frequency using the approximate transverse displacement $w(x, y, t)$ as

$$w(x, y, t) = A(a^2 - x^2 - y^2)^2 e^{i\omega t}, \tag{1}$$

with A and a as the undetermined amplitude and plate radius, respectively. The Rayleigh quotient frequency has the form

$$\omega = \frac{6.33}{a^2}\sqrt{\frac{D_{11} + 0.667(D_{12} + 2D_{66}) + D_{22}}{\rho h}}, \tag{2}$$

where ρ and h are the plate density and thickness, respectively, and D_{11}, D_{22}, D_{12} and D_{66} are the flexural stiffnesses. These stiffnesses are given by

$$D_{11} = \frac{E_x h^3}{12(1 - \nu_{xy}\nu_{yx})}, \qquad D_{22} = \frac{E_y h^3}{12(1 - \nu_{xy}\nu_{ux})},$$
$$D_{66} = \frac{Gh^3}{12}, \qquad D_{12} = \frac{\nu_{yx}E_x h^3}{12(1 - \nu_{xy}\nu_{yx})} = \frac{\nu_{xy}E_y h^3}{12(1 - \nu_{xy}\nu_{yx})}, \tag{3}$$

where E_x, E_y, ν_{xy}, ν_{yx} and G are the orthotropic elastic constants of the material. Rajappa[12] in a short note proposed a solution of this problem by Galerkin's method. However, in carrying out the analysis, he considered an approximate displacement pattern that was a function of the radial coordinate only; i.e. it was independent of the circumferential dependence and thus omitted the coupling with the nonaxisymmetric terms. Thus, his frequency turned out to be exactly the same as that given by eqn (2). A large deflection analysis, which includes transverse shear and rotatory inertia, was presented by Sathymoorthy and Chia[13], who also used Galerkin's method with a one-mode approximation, the spatial dependence of which had the same form as eqn (1). More recently, Luisoni and Laura[14] presented three separate Rayleigh-Ritz analyses to calculate the lowest five frequencies. Two admissible coordinate functions were used to obtain the two so-called axisymmetric modes (no nodal diameter); two coordinate functions for two modes with one nodal diameter and one coordinate function for a mode with two nodal diameters. Since this analysis also does not consider the coupling of the various coordinate functions corresponding to different circumferential mode numbers, the range of applicability will be limited to material properties that have only slight departures from isotropy. The effort in calculating the strain energy integrals even for only two coordinate functions per analysis appears to be quite substantial.

An interior collocation method is employed in our analysis. Interior collocation should be distinguished from boundary collocation, the application of which to plates was popularized by Conway and Leissa[15, 16]. In interior collocation, an approximation sequence of functions is adopted on the basis that it meets the boundary conditions, and point matching of the governing equations in the interior is carried out to generate the matrix eigenvalue problem. These roles are reversed in boundary collocation. This version of interior collocation also differs from its customary application in that overcollocation (i.e. more collocation points than the number of approximation functions) is used toghether with a least-squares error fit. Aside from illustrating the vibrational behavior of such plates, an important purpose of this study is to demonstrate the improvements that are possible with overcollocation. One advantage of collocation over the Rayleigh-Ritz or Galerkin method is that no integrals need to be evaluated, and therefore it is extremely simple to implement the analysis. It is mentioned that the use of a least-squares error fit with boundary collocation was demonstrated on a problem of stress analysis of a plate with single and/or clustered nozzles by Hurlbert *et al.*[17].

In the next section, the least-squares interior collocation method is presented. Then, the details of the application to clamped circular plates with rectilinear orthotropy are elaborated. Numerical examples are given to illustrate the variation of the frequencies with parameterization of the material properties.

LEAST-SQUARES COLLOCATION

Let the governing equation of a boundary value problem for natural vibration analysis be given by

$$D\{w(\mathbf{x})\} = \omega^2 M\{w(\mathbf{x})\}, \tag{4}$$

where $D\{\ \}$ and $M\{\ \}$ are the structural and inertial operators, respectively, and $\mathbf{x}$ denotes the independent spatial variables, the number of which depends on the mathematical dimensionality of the problem. In the interior collocation, a solution set of N approximation functions is chosen, i.e.

$$w(\mathbf{x}) = \sum_{i=1}^{N} A_i W_i(\mathbf{x}), \tag{5}$$

where each $\mathbf{W}_i(\mathbf{x})$ meets the boundary conditions but not necessarily governing equation (4). In the usual application of the interior collocation, it is insisted that the solution

set of N functions satisfy governing equation (4) at N interior stations. Let x_i ($i = 1, 2, \ldots, N$) denote these stations. Then solution form (5) substituted into governing equation (4) and evaluated at these collocation points yields

$$\sum_{j=1}^{N} A_j D\{W_j(\mathbf{x}_i)\} - \omega^2 \sum_{j=1}^{N} A_j M\{W_j(\mathbf{x}_i)\} = 0, \qquad i = 1, 2, \ldots, N. \tag{6}$$

Using the notation

$$k_{ij} = D\{W_i(\mathbf{x}_j)\}, \qquad m_{ij} = M\{W_i(\mathbf{x}_j)\}, \tag{7}$$

eqn (6) can be represented by

$$[\mathbf{K}]\{\mathbf{A}\} - \omega^2[\mathbf{M}]\{\mathbf{A}\} = 0. \tag{8}$$

In algebraic eigenvalue problem (8), [K] and [M] are not necessarily symmetric, which is a major mathematical difference from the Rayleigh-Ritz and Galerkin methods. The solution to (8) provides a set of N^2 frequencies, ω_i^2 values and the right eigenvectors, which represent the synthesis of the modal patterns from the N generalized coordinate A_i values.

The accuracy of this method as well as other approximate methods is obviously tied to the number and quality of approximation functions used, and there is no need for further elaboration at this point. In the collocation method, the locations of the stations are left to the analyst, and there are no prescribed rules for their choice. Two analysts with distinct sets of locations will arrive at different results. Moreover, there is an inherent possibility for "poor" choices of collocation points. A way to overcome this possible deficiency is to use more stations than the number of approximation functions. The enforcement of governing equation (4) at more stations leads to an overdetermined system of equations. The associated matrix form of the collocation equations is rectangular rather than square. This overdetermined system may be solved in terms of a least-squares error fit.

To outline this version of the collocation method, let N denote the number of approximation functions and let R denote the number of collocation stations, with $R > N$. The introduction of these N functions at the R stations can be interpreted as leaving an error at each collocation point, i.e.

$$\sum_{j=1}^{N} A_j D\{W_j(\mathbf{x}_i)\} - \omega^2 \sum_{j=1}^{N} A_j M\{W_j(\mathbf{x}_i)\} = \epsilon_i, \qquad i = 1, 2, \ldots, R \tag{9}$$

or in matrix form

$$[\mathbf{K}]\{\mathbf{A}\} - \omega^2[\mathbf{M}]\{\mathbf{A}\} = \{\boldsymbol{\epsilon}\}. \tag{10}$$

The sum of the square of the pointwise errors is given by

$$\epsilon^2 = \{\boldsymbol{\epsilon}\}^T\{\boldsymbol{\epsilon}\} = \{\mathbf{A}\}^T[\mathbf{KS}]\{\mathbf{A}\} - 2\omega^2\{\mathbf{A}\}^T[\mathbf{KM}]\{\mathbf{A}\} + \omega^4\{\mathbf{A}\}^T[\mathbf{MS}]\{\mathbf{A}\}, \tag{11}$$

where

$$\begin{aligned} [\mathbf{KS}] &= [\mathbf{K}]^T[\mathbf{K}], \qquad [\mathbf{MS}] = [\mathbf{M}]^T[\mathbf{M}] \\ [\mathbf{KM}] &= \frac{1}{2}\left[[\mathbf{K}]^T[\mathbf{M}] + [\mathbf{M}]^T[\mathbf{K}]\right]. \end{aligned} \tag{12}$$

Minimization of ϵ^2 with respect to {A} gives

$$\frac{\partial \epsilon^2}{\partial \{\mathbf{A}\}} = [\mathbf{KS}]\{\mathbf{A}\} - 2\omega^2[\mathbf{KM}]\{\mathbf{A}\} + \omega^4[\mathbf{MS}]\{\mathbf{A}\} = 0. \tag{13}$$

This quadratic eigenvalue problem can be reduced to a first-order system with the following transformation:

$$\{\mathbf{B}\} = \omega^2\{\mathbf{A}\}. \tag{14}$$

Inserting (14) into (13) yields:

$$[\mathbf{KS}]\{\mathbf{A}\} - 2[\mathbf{KM}]\{\mathbf{B}\} + \omega^2[\mathbf{MS}]\{\mathbf{B}\} = 0. \tag{15}$$

Equations (14) and (15) can be combined into the following equation, whose dimensionality is double that of the original system:

$$\begin{bmatrix} 0 & [\mathbf{I}] \\ -[\mathbf{KS}] & 2[\mathbf{KM}] \end{bmatrix} \begin{bmatrix} \{\mathbf{A}\} \\ \{\mathbf{B}\} \end{bmatrix} = \omega^2 \begin{bmatrix} [\mathbf{I}] & 0 \\ 0 & [\mathbf{MS}] \end{bmatrix} \begin{bmatrix} \{\mathbf{A}\} \\ \{\mathbf{B}\} \end{bmatrix}. \tag{16}$$

The solution to (16) should consist of duplicate sets of squared frequencies and their modal combinations of the generalized coordinates, because of the "perfect square" nature of eqn (13). In numerical applications, however, there may not be this duplicity, but rather complex conjugate pairs of solutions. This is due to the fact that the approximation functions are not exact solutions, and there may be slight deficiencies in numerical precision in forming the algebraic system. In all of the examples observed by the authors to date, the imaginary parts of the complex conjugate pairs are very much smaller than the real parts and may be regarded as numerically spurious. The real parts of the solutions approximate the true eigendata very closely for those examples where comparisons with known data are available.

One additional comment can be made. Note that it is possible to introduce a weighting matrix [W] into the least-squares expression (11), i.e.

$$\epsilon^2 = \{\mathbf{g}\}^T[\mathbf{W}]\{\mathbf{g}\}. \tag{17}$$

This would be useful if there were *a priori* information concerning the veracity of certain collocation points. In this article, however, this particular aspect will not be taken up.

PLATE ANALYSIS

To investigate the natural vibrations of a rectilinear orthotropic clamped circular plate of radius a and thickness h, first establish a rectangular Cartesian coordinate system with the origin at the plate's center and let the x-, y-axes run parallel to the natural elastic axes of the material. The governing equation in this coordinate system has the form

$$D_{11}w_{,xxxx} + 2(D_{12} + 2D_{66})w_{,xxyy} + D_{22}w_{,yyyy} - \rho h\omega^2 w = 0. \tag{18}$$

In interior collocation, eqn (18) must be enforced at all the selected stations within the plate. To accommodate the clamped boundary conditions on the plate's circular perimeter, it is more convenient to use approximation functions given in polar coordinates. Hence, the differential operators appearing in (18) should be similarly expressed, i.e. by transformation $[x, y = r\cos(\theta), r\sin(\theta)]$. The algebraic details of this transformation are straightforward, and the expressions for $w_{,xxxx}$, $w_{,xxyy}$, and $w_{,yyyy}$ in terms of r and θ are given in the Appendix.

Owing to twofold structural symmetry, collocation within one-quarter of the plate is sufficient, but four separate cases according to symmetry and antisymmetry conditions about the x- and y-axes must be considered to completely explore the modal behavior. These cases are outlined in Table 1. The forms of the approximation functions can be grouped into two sets depending upon symmetry or antisymmetry conditions

Table 1. Summary of the parameters for the approximation functions

Case	x-y symmetry/ antisymmetry conditions	Solution form eqn	$\cos(n\theta)$ $\sin(n\theta)$	$W_m(r)$	Eqn for $\xi(r)$
1	x sym	(19)	$n = 0$	$m = 1, 3, 5, \ldots$	(23)
	y sym		$n = 2, 4, 6, 8, \ldots$	$m = 1, 2, 3, \ldots$	(24)
2	x sym	(19)	$n = 1$	$m = 2, 4, 6, \ldots$	(23)
	y anti		$n = 3, 5, 7, 9, \ldots$	$m = 1, 2, 3, \ldots$	(24)
3	x anti	(20)	$n = 2, 4, 6, 8, \ldots$	$m = 1, 2, 3, \ldots$	(24)
	y anti				
4	x anti	(20)	$n = 1$	$m = 2, 4, 6, \ldots$	(23)
	y sym		$n = 3, 5, 7, 9, \ldots$	$m = 1, 2, 3, \ldots$	(24)

about the x-axis. They are

$$w(r, \theta) = \sum_n \sum_m A_{mn} W_m(r) \cos(n\theta) \qquad \text{(symmetry)} \tag{19}$$

and

$$w(r, \theta) = \sum_n \sum_m A_{mn} W_m(r) \sin(n\theta) \qquad \text{(antisymmetry)}, \tag{20}$$

where A_{mn} values are the generalized coordinates. For all cases, the radial functions $W_m(r)$ are the beam vibration eigenfunctions with both ends clamped, i.e.

$$W_m(r) = \cosh(\beta_m \xi) - \cos(\beta_m \xi) - \alpha_m[\sinh(\beta_m \xi) - \sin(\beta_m \xi)], \qquad 0 < \xi < 1. \tag{21}$$

The coefficients β_m and α_m may be found in Young and Felgar[18]. They are also recorded in the Appendix for convenience. The radial functions $W_m(r)$ must meet certain conditions at the center of the plate, and these conditions depend on the circumferential mode number n. Moreover, they dictate the form of the relation between variable ξ in (21) and the radial coordinate r. These conditions on the displacement at $r = 0$ (see [19]) are

$$\begin{aligned} n &= 0, \quad & w &= 0 \text{ and } w_{,r} = 0 \\ n &= 1, \quad & w &= 0 \text{ and } w_{,r} = 0 \\ n &\geq 2, \quad & w &= w_{,r} = 0. \end{aligned} \tag{22}$$

For $W_m(r)$ to abide by conditions (22), it is necessary that ξ have one of the two following forms, according to the value of n:

$$\xi = \frac{r + a}{2a}, \qquad \text{for } n = 0 \text{ and } n = 1 \tag{23}$$

$$\xi = \frac{r}{a}, \qquad \text{for } n \geq 2. \tag{24}$$

In Table 1, the appropriate approximation functions for each of the four cases of symmetry/antisymmetry conditions are summarized. For given values of n, the appropriate values of m and the corresponding form for ξ are indicated. It is of interest to note that Case (4) may be obtained from Case (2) by interchanging the roles of D_{11} and D_{22} in governing equation (18).

Collocation simply requires the substitution of eqn (19) or (20) into governing equation (18) expressed in terms of polar coordinates using the proper set of approximation functions as set forth in Table 1.

DISCUSSION OF LEAST-SQUARES COLLOCATION

A very abbreviated evaluation of least-squares collocation was conducted. The axisymmetric vibration of an isotropic plate was considered. In this example, comparisons were made on the usage of both three and four approximation functions with customary collocation and overcollocation. Table 2 summarizes the results in which the number of evenly spaced collocation stations was systematically increased. In all cases, overcollocation results were closer to the exact solution, albeit only slightly for this example. The accuracy of collocation results, like that of all other approximate analysis techniques, depends on the number and the quality of the approximation functions used, which, for this example, was evidently very good with only the leading approximation function. Therefore, the improvement by overcollocation is only nominal.

Table 3 contains the axisymmetric vibration results for the isotropic plate in which the number of collocation stations is equal to or one greater than the number of ap-

Table 2. Comparison of axisymmetric frequency for isotropic plate

	$N = 3$		$N = 4$	
R	K^2	% diff.	K^2	% diff.
3	105.010	0.62	—	—
4	104.706	0.33	103.977	−0.39
5	104.786	0.41	104.421	0.06
6	104.609	0.24	104.536	0.17
7	104.626	0.25	104.617	0.24
8	104.572	0.20	104.630	0.26
9	—	—	104.734	0.35
10	—	—	104.142	−0.21

R, number of evenly spaced collocation points; N, number of approximation functions; ω, $K(1/a^2)(D/\rho h)^{1/2}$ with $K^2_{exact} = 104.363$.

Table 3. Frequencies for axisymmetric vibration with randomly selected collocation stations

N	R^*	Randomly selected collocation points (r)	K^2	% diff.
3	3	0.10, 0.20, 0.30	99.106	−5.09
		0.40, 0.50, 0.60	105.765	1.34
		0.05, 0.10, 0.15	68.801	−34.08
		0.10, 0.20, 0.90	131.88	26.37
		0.70, 0.80, 0.90	−164.59	†
3	4	0.10, 0.20, 0.30, 0.40	99.435	−4.72
		0.50, 0.60, 0.70, 0.80	88.385	−15.31
		0.30, 0.40, 0.50, 0.60	103.105	−1.21
		0.05, 0.10, 0.15, 0.20	98.433	−5.68
		0.40, 0.45, 0.50, 0.55	104.075	−0.28
		0.85, 0.90, 0.95, 0.99	−166.87	†
4	4	0.20, 0.40, 0.60, 0.62	102.989	−1.32
		0.10, 0.20, 0.30, 0.40	101.186	−3.04
		0.60, 0.70, 0.80, 0.90	−175.81	†
		0.40, 0.45, 0.50, 0.60	101.332	−2.90
		0.05, 0.10, 0.90, 0.95	76.727	−26.48
4	5	0.10, 0.20, 0.30, 0.40, 0.50	101.891	−2.67
		0.30, 0.40, 0.50, 0.60, 0.70	101.493	−2.27
		0.50, 0.60, 0.70, 0.80, 0.90	−117.741	†
		0.14, 0.17, 0.50, 0.67, 0.84	110.219	5.61

N, number of approximation functions; R^*, number of random collocation stations; ω, $K(1/a^2)(D/\rho h)^{1/2}$ with $K^2_{exact} = 104.363$.

proximation functions. However, the stations were randomly chosen and are not evenly spaced as before. Poor choices of collocation stations led to results that differ significantly from the exact solution and in certain instances to physically absurd negative squared frequencies. From this example, the obvious recommendation is that evenly spaced stations should be used with collocation. This method for selecting collocation stations was followed in all subsequent examples.

KEVLAR PLATE

As an illustration of a contemporary material, consider a plate made from an aramid cloth-reinforced epoxy composite. The material properties used in this analysis are

$$\begin{aligned} E_x &= E_y = 4 \times 10^3 \text{ ksi} \quad (27.579 \times 10^9 \text{ N/m}^2) \\ G &= 0.5 \times 10^3 \text{ ksi} \quad (3.447 \times 10^9 \text{ N/m}^2) \\ \nu &= 0.14. \end{aligned} \tag{25}$$

These properties are representative of an actual material fabricated from Kevlar, and for convenience in our discussion, it will be referred to as a "Kevlar plate." The weak shear modulus and relatively low Poisson's ratio characterizing Kevlar should allow the plate to behave as if it were composed of an orthogonal grillage of beams. The

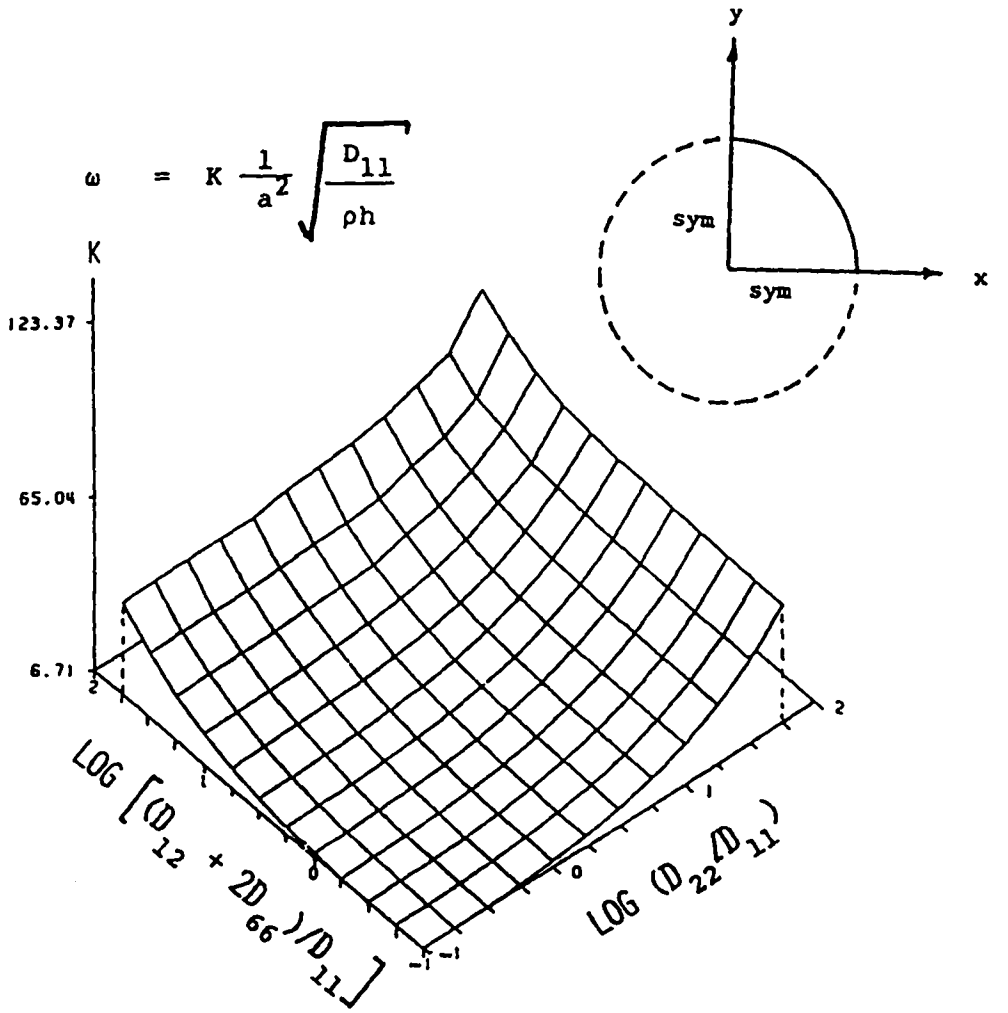

Fig. 1. Frequencies with symmetric/symmetric conditions.

Table 4. Frequencies for Kevlar plate

	K		
M	sym/sym	sym/anti	anti/anti
1	9.4396	19.549	30.368
2	33.433	46.210	63.922
3	35.522	55.465	73.356
4	62.556	88.838	106.50
5	79.491	102.79	132.84

Functions used: sym/sym $n = 0$ with $m = 1, 3$, $n = 2, 4$ with $m = 1, 2, 3$; sym/anti $n = 1$ with $m = 2, 4$, $n = 3, 5$ with $m = 1, 2, 3, 4$; anti/anti $n = 2, 4, 6$ with $m = 1, 2, 3, 4$. ω, $K(1/a^2)(D_{11}/\rho h)^{1/2}$.

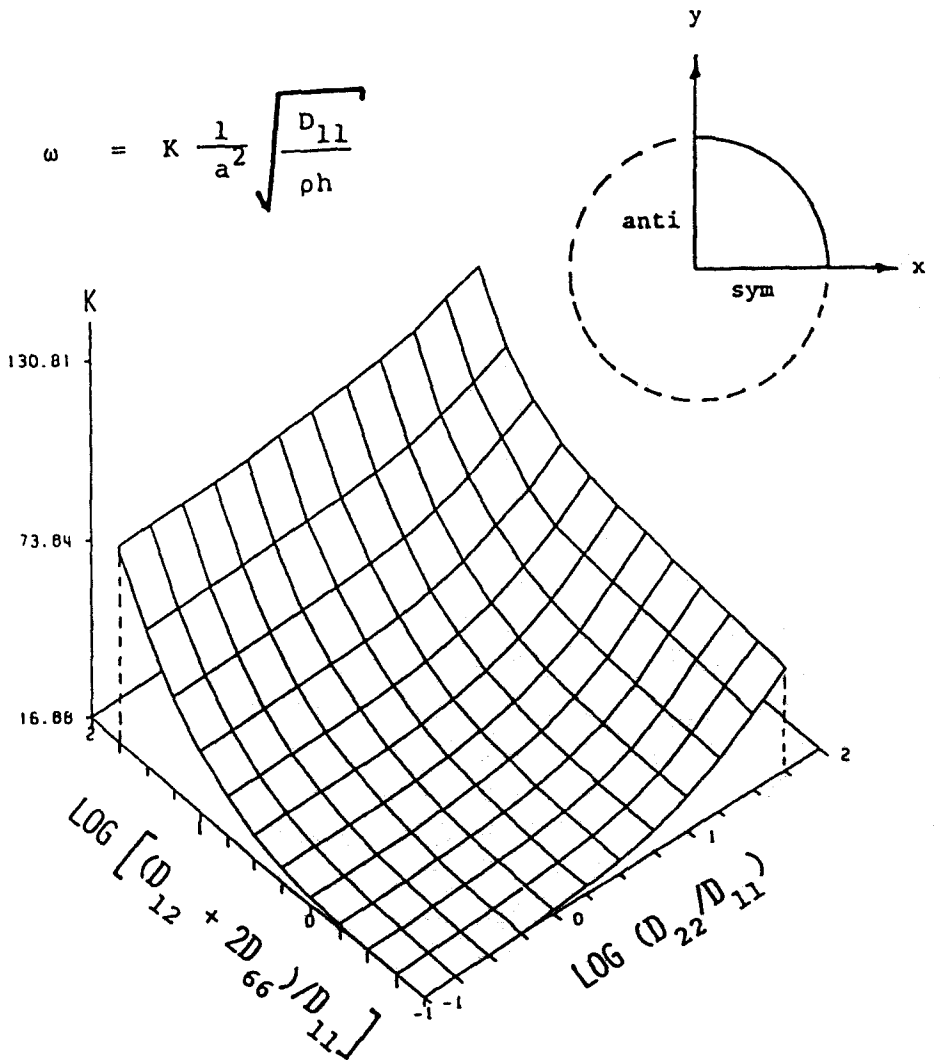

Fig. 2. Frequencies with symmetric/antisymmetric conditions.

Table 5. Comparison of lowest symmetric/symmetric frequencies

Log $[(D_{12}+2D_{66})/D_{11}]$	Log (D_{22}/D_{11})											
	−1.00		−0.50		0.0		0.5		1.0		1.5	
	Present	Eqn (2)	Present	Eqn (2)	Present	Eqn (2)	Present	Eqn (2)	Present	Eqn (2)	Present	Eqn (2)
−1.00	6.7066	6.8372	7.3750	7.4439	9.0418	9.1000	12.929	13.017	20.756	21.058	35.263	36.192
−0.75	6.8644	6.9876	7.5136	7.5822	9.1550	9.2135	13.009	13.097	20.809	21.107	35.295	36.220
−0.50	7.1333	7.2473	7.7535	7.8222	9.3528	9.4120	13.149	13.237	20.901	21.195	35.353	36.271
−0.25	7.5822	7.6875	8.1624	8.2317	9.6942	9.7550	13.394	13.483	21.062	21.349	35.455	36.362
0.0	8.3127	8.4136	8.8410	8.9136	10.272	10.337	13.818	13.910	21.347	21.621	35.627	36.522
0.25	9.4647	9.5697	9.9308	10.012	11.226	11.298	14.544	14.638	21.839	22.097	35.948	36.806
0.50	11.218	11.338	11.615	11.714	12.745	12.830	15.748	15.851	22.680	22.918	36.512	37.305
0.75	13.791	13.939	14.115	14.246	15.066	15.177	17.685	17.804	24.087	24.310	37.465	38.176
1.00	17.451	17.641	17.706	17.885	18.479	18.635	20.680	20.830	26.385	26.606	37.567	39.677
1.25	21.926	22.784	22.688	22.958	23.339	23.562	23.992	25.334	29.719	30.262	41.101	42.216
1.50	28.675	29.813	28.731	29.958	29.182	30.412	30.036	31.804	34.928	35.854	43.530	46.388
1.75	37.823	39.322	38.071	39.432	38.050	39.778	40.487	40.853	43.398	44.079	50.378	53.003
2.00	50.527	52.109	50.547	52.192	50.560	52.454	50.831	53.273	54.021	55.786	61.180	63.075

factor K in the frequency formula

$$\omega = K \frac{1}{a^2} \sqrt{\frac{D_{11}}{\rho h}} \tag{26}$$

is shown in Table 4 for the three classifications of symmetry/antisymmetry conditions. The numbers of approximation functions are also shown in Table 4. It is of interest to note that the lowest isotropic plate modes are for $n = 0, 1, 2$. Because of the lower shear modulus and cross-elasticity coupling, the comparisons of K show the Kelvar plate frequencies to be lower than those for the corresponding isotropic plate, i.e. $n = 0$, 10.216 vs. 9.4396; $n = 1$, 21.26 vs. 19.549; $n = 2$, 34.88 vs. 30.368 (see [1, p. 8] for the isotropic plate results). A comparison for the higher modes is more difficult since the Kelvar results may not have corresponding isotropic plate modal patterns. The higher modes for the Kevlar plate involve coupling of all circumferential modes, which is not the situation for the isotropic plate.

PARAMETRIC STUDY OF ORTHOTROPY

The frequency in the form of the factor K in frequency formula (26) is plotted in Figs 1–3 for the lowest modes for each of the three symmetry/antisymmetry conditions.

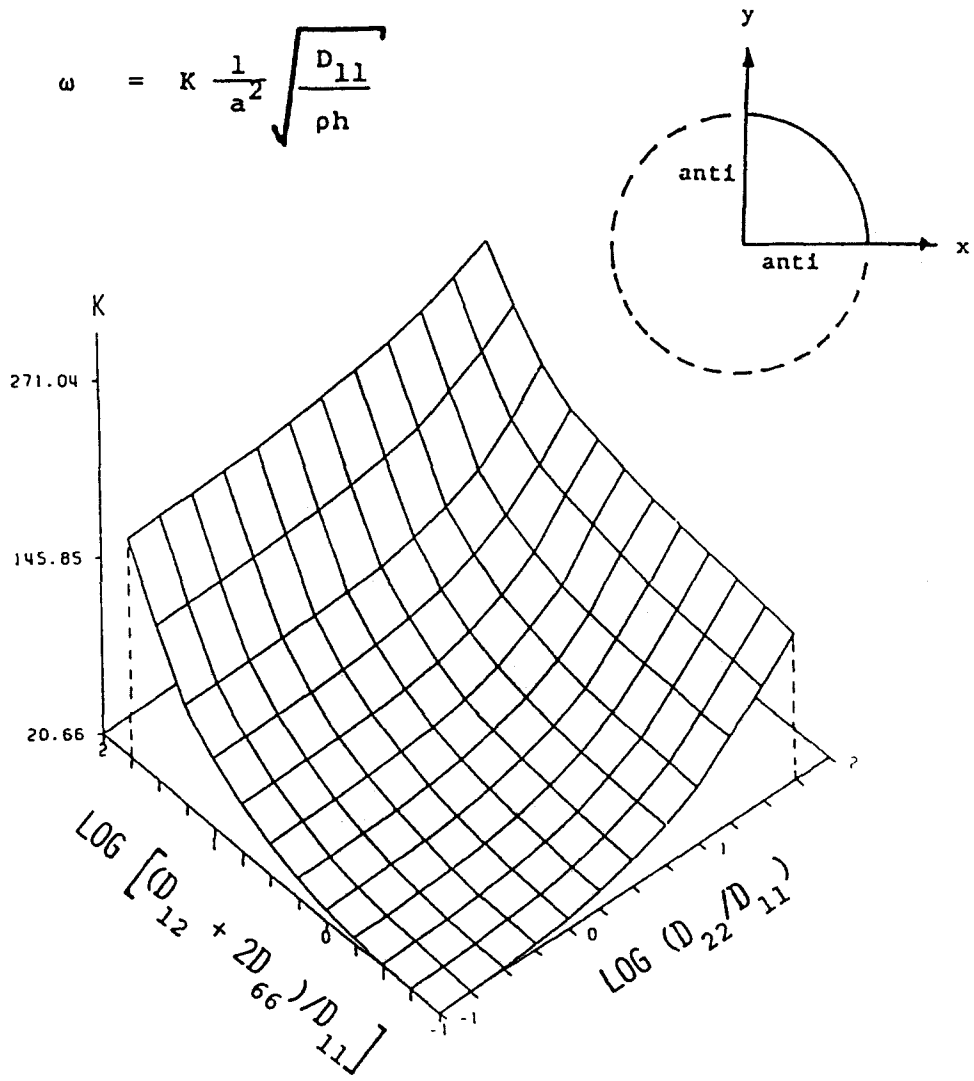

Fig. 3. Frequencies with antisymmetric/antisymmetric conditions.

In these plots, the two axes are the values of D_{22}/D_{11} and $(D_{12} + 2D_{66})/D_{11}$ taken logarithmically. For each case, the lowest three radial modes are combined with the lowest six circumferential modes to form the set of approximation functions. Sufficient collocation stations were used to ensure convergence of the results consistent with the number of approximation functions. The plots show the quantitative variation of the factor K with increasing stiffness ratios D_{22}/D_{11} and $(D_{12} + 2D_{66})/D_{11}$.

It is of interest to compare the present results for the lowest symmetric/symmetric mode with the Rayleigh quotient (2) given by Lekhnitskii[11]. Table 5 contains this comparison for selected values of D_{22}/D_{11} as a function of $(D_{12} + 2D_{66})/D_{11}$, with these ratios tabulated logarithmically. As can be seen, formula (2) is indeed very reliable unless the ratios of the orthotropic properties become excessive.

CONCLUDING REMARKS

A method of least-squares collocation was presented herein and demonstrated for the problem of the vibrations of a clamped circular plate with rectilinear orthotropy. Least-squares collocation has the advantage of ensuring the best possible results consistent with the number and quality of the set of approximation functions used. In applying least-squares collocation, an evenly spaced system of collocation points should be used. This selection method of collocation points may be interpreted as a trapezoidal rule of summation of the least-squares error over the domain of consideration. The method is extremely simple to implement, which is a decided advantage over the Rayleigh-Ritz and Galerkin methods where energy or error integrals must be evaluated analytically before numerical analysis. However, collocation does not provide an upper bound as in Rayleigh-Ritz. If physically good and mathematically complete approximation functions are used, then the results should be reasonably close to their true values.

The problem of the rectilinear orthotropic clamped circular plate illustrated the case of application of least-squares collocation. This problem is ideally suited to collocation, and collocation is more efficient in terms of computational effort (and hence cost) than finite elements. The effect of orthotropy was fully investigated over a wide range of material properties, and these results are shown in graphical form for the lowest modes of vibration for three cases of symmetry/antisymmetry conditions on the two radial edges of one-quarter of the circular plate.

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APPENDIX

The operators $w_{,xxxx}$, $w_{,xxyy}$, $w_{,yyyy}$ in terms of r and θ are tabulated below. Each operator is given by the column of trigonometric coefficients multiplying the polar coordinate derivative on the left.

	$w_{,xxxx}$	$w_{,xxyy}$	$w_{,yyyy}$
$w_{,rrrr}$	c^4	s^2c^2	s^4
$r^{-1}w_{,rrr}$	$6s^2c^2$	$s^4 - 4s^2c^2 + c^4$	$6s^2c^2$
$r^{-2}w_{,rr}$	$3s^2(s^2 - 4c^2)$	$-2s^4 + 11s^2c^2 - 2c^4$	$3c^2(c^2 - 4s^2)$
$r^{-3}w_{,r}$	$-3s^2(s^2 - 4c^2)$	$2s^4 - 11s^2c^2 + 2c^4$	$-3c^2(c^2 - 4s^2)$
$r^{-1}w_{,rr\theta}$	$4sc^3$	$2sc(c^2 - s^2)$	$4s^3c$
$r^{-2}w_{,r\theta}$	$12sc(c^2 - s^2)$	$12sc(s^2 - c^2)$	$12sc(c^2 - s^2)$
$r^{-2}w_{,rr\theta\theta}$	$6s^2c^2$	$s^4 - 4s^2c^2 + c^4$	$6s^2c^2$
$r^{-3}w_{,r\theta}$	$8sc(4s^2 - 3c^2)$	$28sc(c^2 - s^2)$	$8sc(3s^2 - 4c^2)$
$r^{-3}w_{,r\theta\theta}$	$6s^2(s^2 - 4c^2)$	$-4s^4 + 22s^2c^2 - 4c^4$	$6c^2(c^2 - 4s^2)$
$r^{-3}w_{,r\theta\theta\theta}$	$-4s^3c$	$2sc(s^2 - c^2)$	$4sc^3$
$r^{-4}w_{,\theta}$	$24sc(c^2 - s^2)$	$24sc(s^2 - c^2)$	$24sc(c^2 - s^2)$
$r^{-4}w_{,\theta\theta}$	$4s^2(9c^2 - 2s^2)$	$6s^4 - 32s^2c^2 + 6c^4$	$4c^2(9s^2 - 2c^2)$
$r^{-4}w_{,\theta\theta\theta}$	$10s^3c$	$5sc(c^2 - s^2)$	$-10sc^3$
$r^{-4}w_{,\theta\theta\theta\theta}$	s^4	s^2c^2	c^4

$s = \sin(\theta)$; $c = \cos(\theta)$.

The coefficients β_m and α_m in eqn (21) are tabulated below:

m	β_m	α_m
1	4.7300408	0.982502216
2	7.8532046	1.000777311
3	10.9956078	0.999966540
4	14.1371655	1.00000145
5	17.2787596	0.99999937

For $m > 5$: $\beta_m \doteq (2m - 1)\pi/2$; $\alpha_m \doteq 1.0$.